Int. J. Solids Structures, 1974, Vol. 10, pp. 531–553. Pergamon Press. Printed in Gt. Britain.

THE ROLE OF CONVERGENCE IN THE THEORY OF SHELLS‡ §

E. R. de ARANTES e OLIVEIRA†
Universidade Técnica de Lisboa, Instituto Superior Técnico, Lisboa-1, Portugal

(Received 27 November 1972; revised 20 May 1973)

Abstract—Similarly to what happens in the finite element method, the concept of convergence can be used for justifying the use of the virtual work and variational theorems in the derivation of the equations of the theory of shells.

It is proved that the two-dimensional solution becomes more and more near the three-dimensional ones as the thickness tends to zero, provided the relative values of the bending and membrane stiffness coefficients are not changed when the shell becomes thinner and thinner.

Such condition can be respected only if the shell is a generalized one, i.e. if the couple-stresses are not supposed to vanish.

The analysis gives a upper bound to the order of magnitude of the distance between the exact and approximate solution and thus provides a powerful method for testing the efficiency and consistency of any particular theory of shells.

1. INTRODUCTION

Although the theory of shells is usually presented as resulting from the three-dimensional model provided by three-dimensional elasticity, it is well known that a direct approach is possible, i.e., that the shell equations can be established without referring to the three-dimensional theory.

The three-dimensional model is privileged however, not because it is more consistent in itself, but because it is believed to provide a better simulation of the mechanical behaviour of solid bodies. For this reason, and not for any other, it is usually regarded as the fundamental model which generates all the others.

Direct approaches for the derivation of shell equations have thus been disliked by most specialists in the field. Indeed, although they can provide the right equations, they say nothing about the connexion between such equations and the three-dimensional ones.

Mathematicians have therefore preferred to derive the shell equations from the three-dimensional model, by introducing approximative assumptions, and a great deal of research[1, 2] has been inspired by the wish of improving and controlling such approximation procedure.

Energy methods have also been neglected by many researchers because, in the past, they have been presented practically as direct methods, in the sense that the connexion between the solutions they lead to and the corresponding three-dimensional solutions has not been conveniently investigated. In other words, the virtual work principle and variational theorems have been used, but such use has not been justified.

† Professor of Civil Engineering, Technical University of Lisbon.

‡ Paper presented at the 13th International Congress of Theoretical and Applied Mechanics, Moscow, August 1972.

§ The research reported in this paper was supported by Instituto de Alta Cultura (Projecto TLE/4), Lisbon.

The need for a justification was felt more strongly when the energy methods started to be used in the generation of discrete models, with the help of the finite element technique, as it became quite clear that such discrete models were to be discarded if the sequences of approximate discrete solutions obtained by considering successive subdivisions of the body into finite elements with indefinitely decreasing dimensions did not converge to the exact solutions.

The criterion of the convergence to the exact solution was not new however: it has indeed been considered as the fundamental one to be respected by any other approximate method of mathematical analysis like the Ritz method and the finite difference method.

In a previous paper by the author[3], the mathematical theory of structures was presented as a hierarchic collection of models successively generated with the help of two general methods which were justified by theorems of convergence.

In such synthetic vision the theory of shells appears as a two-dimensional approximate model generated from the three-dimensional one, and the convergence concepts and theorems must be used in order that it can be declared as a valid approximation.

The situation is however not quite the same as in the cases considered above. In such cases, indeed, a sequence of approximate solutions must converge to the exact one. In the theory of shells, however, what must be appreciated is the convergence of a sequence of three-dimensional solutions to a two-dimensional solution, i.e. the convergence of a sequence of exact solutions to the approximate one.

The important point indeed is to prove that the two-dimensional solution becomes more and more near the corresponding three-dimensional ones as the shell becomes thinner and thinner, provided the relative values of the bending and membrane stiffnesses are not changed when the thickness tends to zero.

Such condition cannot be respected in a classical shell in which the bending moments result merely from the ordinary stresses distributed in the thickness t, because the bending stiffnesses are then proportional to t^3 and the membrane stiffnesses simply to t. But it can be respected in a generalized shell[4] in which the couple-stresses are not supposed to vanish.

In the present paper, some general concepts and results presented in previous papers are first reminded and the three- and two-dimensional models are then described independently of each other. Two general approaches for deriving the two-dimensional model from the three-dimensional one are finally considered and the application of convergence analysis is exemplified as applied to both approaches.

2. SOME GENERAL CONCEPTS AND RESULTS

As the present paper is essentially concerned with the variational derivation of the theory of shells, a general theory of variational methods will be needed which was developed in another paper by the author[5]. The main topics of such theory will be summarized in this section.

The theory considers a variational problem consisting in the minimization of a given functional F on a space X, and an approximate solution s_a' to such problem obtained from the minimization of functional F on a subspace X' of X. Its main result is inequality (2.12) which provides an upper bound for the distance between the approximate solution, s_a', and the exact solution, s.

Let X be a Banach[6] space and let $\mathscr{F}$ denote a family of continuous functionals on X.

Assume that each functional of the family admits a proper global minimizer in each subset of a certain class of subsets of X, called constrained subsets of X. This means that each constrained subset $C \subset X$ contains an element s such that

$$F(s) < F(c) \tag{2.1}$$

where c denotes any element in C distinct from s.

The constrained subsets are assumed to be homeomorphic to a certain subspace of X and the union of all the constrained subsets is assumed to coincide with the whole space, X.

Let us consider now, for each functional F of the family $\mathscr{F}$, the set of all the proper global minimizers corresponding to all the constrained subsets of X. We call such set the minimizing subset of X corresponding to F and assume the family $\mathscr{F}$ to be such that the union of all the minimizing subsets corresponding to all the different functionals of $\mathscr{F}$ coincides with the whole space, X.

The intersection of each minimizing and each constrained subset is assumed to contain one, and no more than one, element.

Two elements belonging to the same constrained (or minimizing) subset are called isoconstrained (or isominimizing) elements.

Let now B be an operator with domain X and range X', X' denoting a subspace of X. Considering the definition of operator, a unique element in X' must correspond to each element in X, although more than one element in X may correspond to each element in X'. Operator B is assumed:

(i) bounded and continuous,

and such that

(ii) The B-image of any element belonging to X' coincides with the element itself, i.e.

$$B(e) = e \quad \text{if} \quad e \text{ is an element of } X', \tag{2.2}$$

(iii) constrained and minimizing subsets, meeting the same requirements as those in X, can be defined in X', with respect to the same family of functionals, $\mathscr{F}$,

(iv) the B-images of isoconstrained elements in X are isoconstrained elements in X', although not necessarily in X.

The constrained subset C' of X' which contains the B-images of the elements of a given constrained subset C of X is said to correspond to C.

A given minimizing subset D' of X' is said to correspond to a certain minimizing subset D of X if they both correspond to the same functional $F \in \mathscr{F}$.

A second operator, A, can thus be considered, also with domain X and range X', which makes the intersection of each constrained and each minimizing subset of X correspond to the intersection of the corresponding constrained and minimizing subsets of X'.

Operator A is assumed also bounded and continuous, just like B, and also such that

$$A(e) = e \quad \text{if} \quad e \in X'. \tag{2.3}$$

The A-image of an element e of X is called the approximation of e in X'.

We remark that the A-images of any two isominimizing elements of X are isominimizing elements of X'. On the other hand, the A-images of any two isoconstrained elements of X are also isoconstrained elements of X'.

Let now s be an arbitrary element in X, s' the B-image of s and s_a' the A-image of s. s' and s_a' are clearly isoconstrained in X'. The situation is symbolically represented in Fig. 1.

Let C be the constrained subset of X which contains s. Let C' be the corresponding constrained subset of X'. s' and s_a' are both contained in C'.

It is assumed that an element s_a exists in C' such that s_a' is the B-image of s_a.

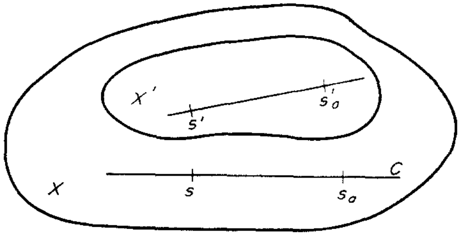

Fig. 1.

Let now s be contained in the minimizing subset of X corresponding to functional F. This means that, by virtue of (2.1),

$$F(s) \leqslant F(s_a). \tag{2.4}$$

On the other hand, as s_a' is the A-image of s, s_a' belongs to the minimizing subset of X' which also corresponds to F, and therefore

$$F(s_a') \leqslant F(s'). \tag{2.5}$$

Let now

$$\delta F = F(s) - F(s') \tag{2.6}$$

$$\delta_a F = F(s_a) - F(s_a'). \tag{2.7}$$

Introducing (2.6) and (2.7) in (2.4), there results

$$F(s') + \delta F - \delta_a F \leqslant F(s_a'). \tag{2.8}$$

Combination of (2.8) with (2.5) yields

$$F(s') + \delta F - \delta_a F \leqslant F(s_a') \leqslant F(s'). \tag{2.9}$$

Therefore,

$$F(s') - F(s_a') \leqslant |\delta F| + |\delta_a F|. \tag{2.10}$$

We assume now that a metric can be introduced in X such that

$$F(s') - F(s_a') = d^2(s, s_a'). \tag{2.11}$$

Combining the triangular inequality with (2.10) and (2.11), there results

$$\mathrm{d}(s, s_a') \leqslant \mathrm{d}(s, s') + \mathrm{d}(s', s_a') \leqslant \mathrm{d}(s, s') + \sqrt{|\delta F| + |\delta_a F|}. \tag{2.12}$$

Inequality (2.12), which gives an upper bound for the distance between any element in X and its approximation in X', will become in the sequel the base of our discussion.

3. THE THREE- AND TWO-DIMENSIONAL MODELS

No difficulty appears in the derivation of the two-dimensional theory if no attempt is made for connecting it with the three-dimensional one.

A complete duality can even be established between the equations of the three- and two-dimensional models and their derivations, if such models are chosen to be the three- and two-dimensional Cosserat theories, and not merely the classical Theory of Elasticity and the theory of shells, which result from the first ones if the couple stresses are assumed to vanish.

Simultaneous independent descriptions of Cosserat's three- and two-dimensional linear theories are presented in this section.

The $\left|\begin{matrix}\text{three}\\ \text{two}\end{matrix}\right|$-dimensional model is referred to a $\left|\begin{matrix}\text{three}\\ \text{two}\end{matrix}\right|$-dimensional domain $\left|\begin{matrix}V\\ S\end{matrix}\right|$ with boundary $\left|\begin{matrix}\Sigma\\ \Gamma\end{matrix}\right|$.

A system of orthogonal co-ordinates $\left|\begin{matrix}\xi_1, \xi_2, \xi_3\\ \xi_1, \xi_2\end{matrix}\right|$ is considered in $\left|\begin{matrix}V\\ S\end{matrix}\right|$.

Let $\left|\begin{matrix}\boldsymbol{\sigma} \text{ and } \boldsymbol{\mu}\\ \mathbf{N} \text{ and } \mathbf{M}\end{matrix}\right|$ (generalized traction vectors) denote the resultant force and moment vectors per unit $\left|\begin{matrix}\text{area}\\ \text{length}\end{matrix}\right|$ of a given $\left|\begin{matrix}\text{surface}\\ \text{line}\end{matrix}\right|$ contained in $\left|\begin{matrix}V\\ S\end{matrix}\right|$. Let $\mathbf{v}$ denote the unit vector normal to such $\left|\begin{matrix}\text{surface}\\ \text{line and tangent to } S\end{matrix}\right|$. Let† $\left|\begin{matrix}\boldsymbol{\sigma}_i \text{ and } \boldsymbol{\mu}_i\\ \mathbf{N}_\alpha \text{ and } \mathbf{M}_\alpha\end{matrix}\right|$ (generalized stress vectors) denote the resultant force and moment vectors associated with the co-ordinate lines.

The equilibrium conditions for an infinitesimal $\left|\begin{matrix}\text{tetrahedron}\\ \text{triangle}\end{matrix}\right|$ bounded by $\left|\begin{matrix}\Sigma' \text{ and three orthogonal co-ordinate surfaces}\\ \Gamma' \text{ and two orthogonal co-ordinate lines}\end{matrix}\right|$ lead to the equilibrium equations‡

$$\boldsymbol{\sigma} = \sum_i \boldsymbol{\sigma}_i v_i \quad\Bigg|\quad \mathbf{N} = \sum_\alpha \mathbf{N}_\alpha v_\alpha \tag{3.1}$$

$$\boldsymbol{\mu} = \sum_i \boldsymbol{\mu}_i v_i . \quad\Bigg|\quad \mathbf{M} = \sum_\alpha \mathbf{M}_\alpha v_\alpha . \tag{3.2}$$

Considering the force equilibrium of an arbitrary fragment of the body, we obtain, with the help of the divergence theorem, the first equilibrium field equation

$$\sum_i \left(H\frac{\boldsymbol{\sigma}_i}{h_i}\right)_{,i} + H\mathbf{f} = \mathbf{0} \quad\Bigg|\quad \sum_\alpha \left(H\frac{\mathbf{N}_\alpha}{h_\alpha}\right)_{,\alpha} + H\mathbf{F} = \mathbf{0} \tag{3.3}$$

where h_i denotes the scale factor along the co-ordinate line i and

$$H = h_1 h_2 h_3 . \qquad H = h_1 h_2 . \tag{3.4}$$

Considering moment equilibrium, we obtain the second equilibrium field equation

$$\sum_i \left[\left(H\frac{\boldsymbol{\mu}_i}{h_i}\right)_{,i} + H\mathbf{a}_i \wedge \boldsymbol{\sigma}_i\right] + H\mathbf{g} = \mathbf{0} \quad\Bigg|\quad \sum_\alpha H\left[\left(\frac{\mathbf{M}_\alpha}{h_\alpha}\right)_{,\alpha} + \mathbf{Ha}_\alpha \wedge \mathbf{N}_\alpha\right] + H\mathbf{G} = \mathbf{0}. \tag{3.5}$$

where $\left|\begin{matrix}\mathbf{f} \text{ and } \mathbf{g}\\ \mathbf{F} \text{ and } \mathbf{G}\end{matrix}\right|$ are the generalized body force and moment densities on $\left|\begin{matrix}V\\ S\end{matrix}\right|$.

The strain–displacement equations

$$\mathbf{e}_i = \frac{\mathbf{u}_{,i}}{h_i} + \mathbf{a}_i \wedge \boldsymbol{\theta} \quad\Bigg|\quad \mathbf{E}_\alpha = \frac{\mathbf{U}_{,\alpha}}{h_\alpha} + \mathbf{a}_\alpha \wedge \boldsymbol{\Theta} \tag{3.6}$$

$$\mathbf{k}_i = \frac{\boldsymbol{\theta}_{,i}}{h_i} \quad\Bigg|\quad \mathbf{K}_\alpha = \frac{\boldsymbol{\Theta}_\alpha}{h_\alpha} \tag{3.7}$$

† Latin indices are supposed to take-up values 1, 2 or 3, while Greek indices can take-up the values 1 and 2.

‡ $\left|\begin{matrix}\text{Three}\\ \text{two}\end{matrix}\right|$-dimensional equations are written at $\left|\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right|$.

can be established by combining the equilibrium equations (3.1), (3.2), (3.3) and (3.5) with equation

$$\left.\begin{aligned} &\int_{v'} \sum_i (\boldsymbol{\sigma}_i \,.\, \mathbf{e}_i + \boldsymbol{\mu}_i \,.\, \mathbf{k}_i)\, \mathrm{d}V \\ &= \int_{v'} (\mathbf{f} \,.\, \mathbf{u} + \mathbf{g} \,.\, \boldsymbol{\theta})\, \mathrm{d}V \\ &\quad + \int_{\Sigma'} (\boldsymbol{\sigma} \,.\, \mathbf{u} + \boldsymbol{\mu} \,.\, \boldsymbol{\theta})\, \mathrm{d}\Sigma \end{aligned}\right| \begin{aligned} &\int_{S'} \sum_\alpha (\mathbf{N}_\alpha \,.\, \mathbf{E}_\alpha + \mathbf{M}_\alpha \,.\, \mathbf{K}_\alpha)\, \mathrm{d}S \\ &= \int_{S'} (\mathbf{F} \,.\, \mathbf{U} + \mathbf{G} \,.\, \boldsymbol{\Theta})\, \mathrm{d}S \\ &\quad + \int_{\Gamma'} (\mathbf{N} \,.\, \mathbf{U} + \mathbf{M} \,.\, \boldsymbol{\Theta})\, \mathrm{d}\Gamma \end{aligned} \tag{3.8}$$

which expresses the work principle written for an arbitrary fragment $\left|\begin{matrix} V' \\ S' \end{matrix}\right|$, with boundary $\left|\begin{matrix} \Sigma' \\ \Gamma' \end{matrix}\right|$ of the body. The vector fields involved in (3.8) are of course supposed to admit the derivatives involved in (3.3), (3.5), (3.6) and (3.7).

We could also start from (3.6) and (3.7) and derive the equilibrium equations with the help of (3.8).

Constitutive equations can be found now by making the strain energy density on $\left|\begin{matrix} V,\ W_V \\ S,\ W_S \end{matrix}\right|$, an exclusive function of vectors $\left|\begin{matrix} \mathbf{e}_i \text{ and } \mathbf{k}_i \\ \mathbf{E}_\alpha \text{ and } \mathbf{K}_\alpha \end{matrix}\right|$.

Let indeed the components of vectors $\left|\begin{matrix} \boldsymbol{\sigma}_i,\ \boldsymbol{\mu}_i,\ \mathbf{e}_i \text{ and } \mathbf{k}_i \\ \mathbf{N}_\alpha,\ \mathbf{M}_\alpha,\ \mathbf{E}_\alpha \text{ and } \mathbf{K}_\alpha \end{matrix}\right|$ be denoted by $\left|\begin{matrix} \sigma_{ij},\ \mu_{ij},\ e_{ij} \text{ and } k_{ij} \\ \mathbf{N}_{\alpha j},\ M_{\alpha j},\ E_{\alpha j} \text{ and } K_{\alpha j} \end{matrix}\right|$.

The stress–strain equations have then the form

$$\sigma_{ji} = \frac{\partial W_V}{\partial e_{ji}} \quad \Bigg| \quad N_{\alpha i} = \frac{\partial W_S}{\partial E_{\alpha i}} \tag{3.9}$$

$$\mu_{ji} = \frac{\partial W_V}{\partial k_{ji}} \quad \Bigg| \quad M_{\alpha i} = \frac{\partial W_S}{\partial K_{\alpha i}} \tag{3.10}$$

It is easy to conclude from (3.1) and (3.2), and from the invariance of internal work that $\left|\begin{matrix} \sigma_{ij},\ \mu_{ij},\ e_{ij} \text{ and } k_{ij} \\ N_{\alpha\beta},\ M_{\alpha\beta},\ E_{\alpha\beta} \text{ and } K_{\alpha\beta} \end{matrix}\right|$ are components of second-order $\left|\begin{matrix} \text{three-dimensional} \\ \text{two-dimensional} \end{matrix}\right|$ tensors [and that $N_{\alpha 3}$, $M_{\alpha 3}$, $E_{\alpha 3}$ and $K_{\alpha 3}$ are components of vectors].†

If isotropy is assumed on $\left|\begin{matrix} V \\ S \end{matrix}\right|$, then, $\left|\begin{matrix} W_V \\ W_S \end{matrix}\right|$ must be an exclusive function of the fundamental invariants of such strain tensors and vectors.

As linearity is assumed, the only invariants which can be involved are the first and second invariants of the symmetric parts of tensor $\left|\begin{matrix} e_{ij} \\ E_{\alpha\beta} \end{matrix}\right|$, the second invariant (the first always vanishes) of the skew-symmetric part of the same tensor, [the invariant of vector $E_{\alpha 3}$ (its length)], i.e.

† The parts of the text which concern exclusively the two-dimensional model are within brackets.

$$e = e_{11} + e_{22} + e_{33}$$

$$e_s' = \begin{vmatrix} e_{22} & \frac{1}{2}(e_{23} + e_{32}) \\ \frac{1}{2}(e_{23} + e_{32}) & e_{33} \end{vmatrix} + \cdots$$

$$e_a' = \begin{vmatrix} 0 & \frac{1}{2}(e_{23} - e_{32}) \\ \frac{1}{2}(e_{32} - e_{23}) & 0 \end{vmatrix} + \cdots$$

$$E = E_{11} + E_{22} \tag{3.11}$$

$$E_s' = \begin{vmatrix} E_{11} & \frac{1}{2}(E_{12} + E_{21}) \\ \frac{1}{2}(E_{12} + E_{21}) & E_{22} \end{vmatrix} \tag{3.12}$$

$$E_a' = \begin{vmatrix} 0 & \frac{1}{2}(E_{12} - E_{21}) \\ \frac{1}{2}(E_{21} - E_{12}) & 0 \end{vmatrix} \tag{3.13}$$

$$E_3 = E_{13}{}^2 + E_{23}{}^2 \tag{3.14}$$

and the corresponding invariants $\begin{vmatrix} k, k_s' \text{ and } k_a' \\ K, K_s' \text{ and } K_a' \end{vmatrix}$ of the curvatures $\begin{vmatrix} k_{ij} \\ K_{\alpha j} \end{vmatrix}$.

The strain energy density has thus the form

$$W_V = \tfrac{1}{2}(\lambda + 2\mu)e^2 - 2\mu e_s' - 2\rho e_a' + \tfrac{1}{2}(\phi + 2\psi)k^2 - 2\psi k_s' - 2\chi k_a' + \pi e k$$

$$W_s = \tfrac{1}{2}(\bar{\lambda} + 2\bar{\mu})E^2 - 2\bar{\mu}E_s' - 2\bar{\rho}E_a' + \bar{\nu}E_3 + \tfrac{1}{2}(\bar{\phi} + 2\bar{\psi})K^2 - 2\bar{\psi}K_s' - 2\bar{\chi}K_a' + \bar{\tau}k_3 + \bar{\pi}EK \tag{3.15}$$

The term proportional to $\begin{vmatrix} ek \\ EK \end{vmatrix}$ disappears however if it is assumed that no changes of curvature are possible when the components of vectors $\begin{vmatrix} \boldsymbol{\mu}_i \\ \mathbf{M}_\alpha \end{vmatrix}$ vanish everywhere.

The stress–strain equations for the linear case become then†

$$\sigma_{ij} = \mu(e_{ij} + e_{ji}) + \rho(e_{ij} - e_{ji}) + \lambda e \delta_{ij}$$

$$N_{\alpha\beta} = \bar{\mu}e(E_{\alpha\beta} + E_{\beta\alpha}) + \bar{\rho}(E_{\alpha\beta} - E_{\beta\alpha}) + \bar{\lambda}E\delta_{\alpha\beta} \tag{3.16}$$

$$N_{\alpha 3} = 2\bar{\nu}E_{\alpha 3} \tag{3.17}$$

$$\mu_{ij} = \psi(k_{ij} + k_{ji}) + \chi(k_{ij} - k_{ji}) + \phi k \delta_{ij}$$

$$M_{\alpha\beta} = \bar{\psi}(K_{\alpha\beta} + K_{\beta\alpha}) + \bar{\chi}(K_{\alpha\beta} - K_{\beta\alpha}) + \bar{\phi}K\delta_{\alpha\beta} \tag{3.18}$$

$$M_{\alpha 3} = 2\bar{\tau}K_{\alpha 3} \tag{3.19}$$

Equations (3.1), (3.2), (3.5), (3.16–19) must be supplemented by the boundary conditions

$$\left.\begin{matrix} \boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}} \\ \boldsymbol{\mu} = \bar{\boldsymbol{\mu}} \end{matrix}\right\} \text{ on } \Sigma_1 \qquad \left.\begin{matrix} \mathbf{N} = \bar{\mathbf{N}} \\ \mathbf{M} = \bar{\mathbf{M}} \end{matrix}\right\} \text{ on } \Gamma_1 \tag{3.20}$$

$$\left.\begin{matrix} \mathbf{u} = \bar{\mathbf{u}} \\ \boldsymbol{\theta} = \bar{\boldsymbol{\theta}} \end{matrix}\right\} \text{ on } \Sigma_2 . \qquad \left.\begin{matrix} \mathbf{U} = \bar{\mathbf{U}} \\ \boldsymbol{\Theta} = \bar{\boldsymbol{\Theta}} \end{matrix}\right\} \text{ on } \Gamma_2 \tag{3.21}$$

In the current literature, the state of the structure free from external forces is usually taken as the underformed state of the structure, with respect to which strains and displacements are measured. If this is done, and initial stresses are present, the stress–strain equations cease to be homogeneous while the strain–displacement equations are always homogeneous.

† [Components $M_{\alpha\beta}$ and $K_{\alpha\beta}$ are usually replaced by $M'_{\alpha\beta}$ and $K'_{\alpha\beta}$ related to $M_{\alpha\beta}$ and $K_{\alpha\beta}$ by

$$M_{\alpha\beta}' = \mathbf{M}_\alpha \cdot \mathbf{a}_3 \wedge \mathbf{a}_\beta$$
$$K_{\alpha\beta}' = \mathbf{K}_\alpha \cdot \mathbf{a}_3 \wedge \mathbf{a}_\beta$$

In terms of such magnitudes, equation (3.18) becomes

$$M'_{\alpha\beta} = 2\bar{\psi}K'_{\alpha\beta} + {}^{-}(K'_{\alpha\beta} - K'_{\beta\alpha}) + (\bar{\chi} - \bar{\psi})(K'_{11} + K'_{22})\delta_{\alpha\beta}.]$$

A different practice is followed however in the present paper and in other papers by the author[3, 7, 8]. According to such practice, the stress–strain equations are kept always homogeneous, i.e. the strains are assumed to vanish always with the stresses. Consequently, the stress–strain equations become inhomogeneous, i.e. vanishing strains (more precisely, generalized strains) do not correspond to vanishing displacements (more precisely, generalized displacements).

We have thus, instead of (3.6) and (3.7),

$$\mathbf{e}_i = \frac{\mathbf{u}_{,i}}{h_i} + \mathbf{a}_i \wedge \boldsymbol{\theta} + \mathbf{e}_i^0 \qquad \mathbf{E}_\alpha = \frac{\mathbf{U}_{,\alpha}}{h_\alpha} + \mathbf{a}_\alpha \wedge \boldsymbol{\Theta} + \mathbf{E}_\alpha^0 \tag{3.22}$$

$$\mathbf{k}_i = \frac{\boldsymbol{\theta}_{,i}}{h_i} + \mathbf{k}_i^0 \qquad K_\alpha = \frac{\boldsymbol{\Theta}_{,\alpha}}{h_\alpha} + \mathbf{K}_\alpha^0 \tag{3.23}$$

The intial strains $\left|\begin{matrix} \mathbf{e}_i^0 & \text{and} & \mathbf{k}_i^0 \\ \mathbf{E}_\alpha^0 & \text{and} & \mathbf{K}_\alpha^0 \end{matrix}\right|$ correspond to the initial stresses $\left|\begin{matrix} \boldsymbol{\sigma}_i^0 & \text{and} & \boldsymbol{\mu}_i^0 \\ \mathbf{N}_\alpha^0 & \text{and} & \mathbf{M}_\alpha^0 \end{matrix}\right|$ through the stress–strain equations (3.16–19). The initial stresses are the stresses which appear in the body when no external forces are acting.

In the sequel, generalized† stresses and strains will be assumed always connected by the stress–strain equations. The association of a strain field and a stress field connected in this way will be termed simply a field. We remark that the generalized displacements corresponding to a given field are determined, except for a rigid-body motion.

A field is said to be compatible, with respect to given initial strains $\left|\begin{matrix} \mathbf{e}_i^0 & \text{and} & \mathbf{k}_i^0 \\ \mathbf{E}_\alpha^0 & \text{and} & \mathbf{K}_\alpha^0 \end{matrix}\right|$ and given displacements prescribed on the boundary, if equations (3.21–23) are satisfied. A field is said to be equilibrated, with respect to given body forces and given forces applied on the boundary, if the satisfied equations are (3.3), (3.5) and (3.20). The stress–strain equations are of course supposed to be satisfied in any case.

A field which is simultaneously a compatible and an equilibrated one is the exact solution with respect to given initial strains, body forces and boundaries conditions. Kirchhoff's theorem states the uniqueness of such solution.

The total potential and the total complementary energy theorems can easily be deduced from the preceeding equations.

The first one states that the exact solution, with respect to given body forces, initial strains and boundary conditions, minimizes the total potential energy.

$$T_3 = \int_V W_V \, \mathrm{d}V - \int_V (\bar{\mathbf{f}} \cdot \mathbf{u} + \bar{\mathbf{g}} \cdot \boldsymbol{\theta}) \mathrm{d}V - \int_{\Sigma_1} (\bar{\boldsymbol{\sigma}} \cdot \mathbf{u} + \bar{\boldsymbol{\mu}} \cdot \boldsymbol{\theta}) \mathrm{d}\Sigma \qquad T_2 = \int_S W_S \, \mathrm{d}S - \int_S (\bar{\mathbf{F}} \cdot \mathbf{U} + \bar{\mathbf{G}} \cdot \boldsymbol{\Theta}) \mathrm{d}S - \int_{\Gamma_1} (\bar{\mathbf{N}} \cdot \mathbf{U} + \bar{\mathbf{M}} \cdot \boldsymbol{\Theta}) \mathrm{d}\Gamma \tag{3.24}$$

on the set of the compatible fields.

The second one states that the exact solution with respect to ..., minimizes the total complementary energy.

† The word "generalized" will be omitted in the sequel whenever the clarity of the text is not affected.

$$T_3^* = \int_V W_V^* \mathrm{d}V - \int_V \sum_i (\boldsymbol{\sigma}_i \cdot \bar{\mathbf{e}}_i^0 + \boldsymbol{\mu}_i \cdot \bar{\mathbf{k}}_i^0)\mathrm{d}V - \int_{\Sigma_2} (\boldsymbol{\sigma} \cdot \bar{\mathbf{u}} + \boldsymbol{\mu} \cdot \bar{\boldsymbol{\theta}})\mathrm{d}\Sigma$$

$$T_2^* = \int_S W_S^* \mathrm{d}S - \int_S \sum_\alpha (\mathbf{N}_\alpha \cdot \bar{\mathbf{E}}_\alpha^0 + \mathbf{M}_\alpha \cdot \bar{\mathbf{K}}_\alpha^0)\mathrm{d}S - \int_{\Gamma_2} (\mathbf{N} \cdot \bar{\mathbf{U}} + \mathbf{M} \cdot \bar{\mathbf{H}})\mathrm{d}\Gamma \tag{3.25}$$

on the set of the equilibrated fields.

We remember that the complementary energy $\left|\begin{matrix} W_V^* \\ W_S^* \end{matrix}\right|$ equals the strain energy $\left|\begin{matrix} W_V \\ W_S \end{matrix}\right|$ because linearity and homogeneity of the stress–strain equations have been assumed.

The fact that T and T^* are minimized and not simply made stationary is connected to the stability assumption, according to which $\left|\begin{matrix} W_V \\ W_S \end{matrix}\right|$ is a positive definite function of the strains.

If linearity is assumed, as indeed it was, it can be shown[7, 8] that

$$T(c) - T(s) = U(c - s) = U^*(c - s) \tag{3.26}$$

$$T^*(e) - T^*(s) = U^*(e - s) = U(e - s) \tag{3.27}$$

where c is any compatible field and e any equilibrated field. s is the exact solution. The total potential and complementary energy theorems immediately result from (3.26) and (3.27). They are true however even if the stress–strain equations are non-linear.

4. THE TWO-DIMENSIONAL MODEL AS DERIVED FROM THE THREE-DIMENSIONAL ONE

The set of the three-dimensional elastic fields forms a linear space which will be denoted by X. So does the set of the two-dimensional fields, which will be denoted by Y. The addition of two elements in X or Y corresponds to the addition of the corresponding stress and strain components.

Isocompatible and isoequilibrated subsets can be considered both in X and Y. The first contain the fields which compatibilize the same incompatibilities.† The second contain the fields which equilibrate the same external forces.‡

The $\left|\begin{matrix} \text{isocompatible} \\ \text{isoequilibrated} \end{matrix}\right|$ subsets may be taken as constrained subsets of X, if the set of the total $\left|\begin{matrix} \text{potential} \\ \text{complementary} \end{matrix}\right|$ energies corresponding to all possible systems of $\left|\begin{matrix} \text{external forces} \\ \text{incompatibilities} \end{matrix}\right|$ is taken for family of continuous functionals. The isominimizing subsets of X are then the $\left|\begin{matrix} \text{isoequilibrated} \\ \text{isocompatible} \end{matrix}\right|$ subsets. The same can be said about Y.

From the two different possibilities considered, two different methods arise for the generation of the two-dimensional model from the three-dimensional one. They will be

† A system of incompatibilities is defined by an initial strain field and a system of displacements prescribed on Σ_2 or Γ_2.

‡ An inner-product definition can be introduced in the linear case such that the isocompatible and isoequilibrated subsets become orthogonal[8]. Such orthogonality will not be used however in the present paper.

called the potential energy method and the complementary energy method and will be simultaneously described in order that their duality becomes more striking.

The three-dimensional body which our discussion basically considers is what we call a shell. Such body can be analyzed both by the three- and two-dimensional theory, this last being referred to the middle-surface of the shell.

In the $\left|\begin{array}{c}\text{potential}\\ \text{complementary}\end{array}\right|$ energy approach the two-dimensional generalized $\left|\begin{array}{c}\text{strains and displacements}\\ \text{stresses and tractions}\end{array}\right|$ are defined in terms of the corresponding three-dimensional magnitudes in such a way that the two-dimensional $\left|\begin{array}{c}\text{strain–displacement}\\ \text{equilibrium}\end{array}\right|$ equations and the two-dimensional boundary conditions on $\left|\begin{array}{c}\Gamma_2\\ \Gamma_1\end{array}\right|$ become exact in the frame of the three-dimensional theory. Such is for instance what happens if the three-dimensional generalized $\left|\begin{array}{c}\text{strains and displacements measured on the middle surface}\\ \text{stress and traction resultants and couples taken over the thickness}\end{array}\right|$ of the shell are taken as the two-dimensional generalized $\left|\begin{array}{c}\text{strains and displacements}\\ \text{stresses and tractions}\end{array}\right|$. We admit therefore†

$$\mathbf{E}_\alpha = (\mathbf{e}_\alpha)_{\xi_3=0} \qquad\bigg|\qquad \mathbf{N}_\alpha = \int_{-t/2}^{t/2} \lambda_{\bar{\alpha}}\,\boldsymbol{\sigma}_\alpha\,\mathrm{d}\xi_3 \tag{4.1}$$

$$\mathbf{K}_\alpha = (\mathbf{k}_\alpha)_{\xi_3=0} \qquad\bigg|\qquad \mathbf{M}_\alpha = \int_{-t/2}^{t/2} \lambda_{\bar{\alpha}}(\mathbf{a}_3\,\xi_3 \wedge \boldsymbol{\sigma}_\alpha + \boldsymbol{\mu}_\alpha)\,\mathrm{d}\xi_3 \tag{4.2}$$

$$\mathbf{U}_i = (\mathbf{u}_i)_{\xi_3=0} \qquad\bigg|\qquad \mathbf{N} = \int_{-t/2}^{t/2} \lambda\boldsymbol{\sigma}\,\mathrm{d}\xi_3 \tag{4.3}$$

$$\boldsymbol{\Theta}_i = (\boldsymbol{\theta}_i)_{\xi_3=0} \qquad\bigg|\qquad \mathbf{M} = \int_{-t/2}^{t/2} \lambda(\mathbf{a}_3\,\xi_3 \wedge \boldsymbol{\sigma} + \boldsymbol{\mu})\,\mathrm{d}\xi_3 \tag{4.4}$$

and

$$\mathbf{E}_\alpha{}^0 = (\mathbf{e}_\alpha{}^0)_{\xi_3=0} \qquad\bigg|\qquad \mathbf{F} = \int_{-t/2}^{t/2} \lambda_1\lambda_2\,\mathbf{f}\,\mathrm{d}\xi_3 \tag{4.5}$$

$$\mathbf{K}_\alpha{}^0 = (\mathbf{k}_\alpha{}^0)_{\xi_3=0} \qquad\bigg|\qquad \mathbf{G} = \int_{-t/2}^{t/2} \lambda_1\lambda_2(\mathbf{a}_3\,\xi_3 \wedge \mathbf{f} + \mathbf{g})\,\mathrm{d}\xi_3 \tag{4.6}$$

In the preceeding equations, $\lambda_{\bar{\alpha}}$‡ denotes the ratio between the scale factor $h_{\bar{\alpha}}$ at an arbitrary point of the shell and the scale factor $h_{\bar{\alpha}}$ at the point of S with the same coordinates ξ_1 and ξ_2. λ plays the same role, on any surface with an analytical expression of the type $f(\xi_1, \xi_2) = 0$, as magnitudes λ_α on the coordinate surfaces.

Equations (4.1–4.4) introduce a correspondence between X and Y. By other words, a linear bounded operator B_1 is introduced with domain X and range Y, such that any two elements which are isoconstrained in X are isoconstrained in Y.

Assume now that a new linear operator B_2 is introduced with domain Y and range X', X' being a subset of X. Such operator is assumed to have an inverse, i.e. to introduce a one-to-one correspondence between Y and X'.

† From now on, the equations at $\left|\begin{array}{c}\text{left}\\ \text{right}\end{array}\right|$ concern the $\left|\begin{array}{c}\text{potential}\\ \text{complementary}\end{array}\right|$ energy approach.

‡ $\bar{\alpha}$ is equal to 1 when α is equal to 2 and equal to 2 when α is equal to 1.

Let f be an arbitrary element of X, g its B_1-image in Y and f' the B_2-image of g in X'. We can write

$$g = B_1(f) \tag{4.7}$$

$$f' = B_2(g) = B_2[B_1(f)] = B(f). \tag{4.8}$$

The product of the two operators B_1 and B_2 is thus a new linear bounded operator, B, with domain X and range $X' \subset X$.

The $\left|\begin{array}{c}\text{potential}\\ \text{complementary}\end{array}\right|$ energy method postulates the invariance of the $\left|\begin{array}{c}\text{potential}\\ \text{complementary}\end{array}\right|$ energies. This means that

$$U_3(B_2(g)) = U_2(g) \quad \Big| \quad U_3{}^*(B_2(g)) = U_2{}^*(g) \tag{4.9}$$

$$T_3(B_2(g)) = T_2(g) \quad \Big| \quad T_3{}^*(B_2(g)) = T_2{}^*(g) \tag{4.10}$$

where $\left|\begin{array}{ccc}U_3 & \text{and} & T_3\\ U_3{}^* & \text{and} & T_3{}^*\end{array}\right|$ denote the $\left|\begin{array}{c}\text{potential}\\ \text{complementary}\end{array}\right|$ energies computed in the frame of the three-dimensional theory, and $\left|\begin{array}{ccc}U_2 & \text{and} & T_2\\ U_2{}^* & \text{and} & T_2{}^*\end{array}\right|$ denote the same magnitude in the frame of the two-dimensional theory.

Equation (4.9) permits to express the two-dimensional elastic coefficients in terms of the three-dimensional ones. Indeed, (4.9) implies

$$\int_S W_S \, \mathrm{d}S = \int_V W_V \, \mathrm{d}V \quad \Big| \quad \int_S W_S{}^* \, \mathrm{d}S = \int_V W_V{}^* \, \mathrm{d}V \tag{4.11}$$

and thus

$$W_S = \int_{-t/2}^{t/2} \lambda_1 \lambda_2 \, W_V \, \mathrm{d}\xi_3 \quad \Big| \quad W_S{}^* = \int_{-t/2}^{t/2} \lambda_1 \lambda_2 \, W_V{}^* \, \mathrm{d}\xi_3 \tag{4.12}$$

and this permits to write, with the help of (3.9) and (3.10), the stress–strain equation in terms of the three-dimensional coefficients. Comparison with the same equations expressed in terms of the two-dimensional coefficients yields the relations between these last and the three-dimensional ones.

Equation (4.10) permits to express $\left|\begin{array}{c}\overline{\mathbf{F}}, \overline{\mathbf{G}}, \overline{\mathbf{N}} \text{ and } \overline{\mathbf{M}}\\ \overline{\mathbf{E}}^0, \overline{\mathbf{K}}^0, \overline{\mathbf{U}} \text{ and } \overline{\mathbf{\Theta}}\end{array}\right|$ in terms of the corresponding generating magnitudes.

By virtue of (4.10) the B_2-images of the constrained and minimizing subsets of Y are the constrained and minimizing subsets of X'.

An obvious consequence is that the B_2-images of elements in Y are isoconstrained elements in X', and as the B_1-images of isoconstrained elements in X are isoconstrained in Y, we conclude that the B-images of isoconstrained elements in X are isoconstrained elements in X'.

If we assume now that the B-image of any element belonging to X' coincides with the element itself, all the requirements made in Section 2 for operator B are respected and all the results established in this section can be applied.

The only point which is still missing is the introduction of a metric in X such that (2.11) holds. This will be achieved if the distance between two fields is defined as the square-root of the $\left|\begin{array}{c}\text{strain}\\ \text{complementary}\end{array}\right|$ energy of their difference, i.e.

$$d(f_1, f_2) = \sqrt{(U_{f_1 - f_2})} = \sqrt{U^*_{f_1 - f_2}} \tag{4.13}$$

Then indeed (3.26) and (3.27) imply (2.11).

Consider now equation (2.12). According to the definition of distance,

$$d(s, s') = \sqrt{\left[\frac{1}{2}\int_V \sum_i (\delta\boldsymbol{\sigma}_i \cdot \delta\mathbf{e}_i + \delta\mathbf{k}_i \cdot \delta\boldsymbol{\mu}_i)\,\mathrm{d}V\right]} \tag{4.14}$$

where δ denotes the variation of a magnitude from s to s'.

On the other hand,

$$\delta F = \delta T_3(s) = \delta U_3 - \int_V (\bar{\mathbf{f}} \cdot \delta\mathbf{u} + \bar{\mathbf{g}} \cdot \delta\boldsymbol{\theta})\,\mathrm{d}V - \int_{\Sigma_1} (\bar{\boldsymbol{\sigma}} \cdot \delta\mathbf{u} + \bar{\boldsymbol{\mu}} \cdot \delta\boldsymbol{\theta})\,\mathrm{d}\Sigma$$

$$\delta F = \delta T_3{}^*(s) = \delta U_3{}^* - \int_V \sum_i (\delta\boldsymbol{\sigma}_i \cdot \bar{\mathbf{e}}_i{}^0 + \delta\boldsymbol{\mu}_i \cdot \bar{\mathbf{k}}_i)\,\mathrm{d}V - \int_{\Sigma_2} (\delta\boldsymbol{\sigma} \cdot \delta\bar{\mathbf{u}} + \delta\boldsymbol{\mu} \cdot \bar{\boldsymbol{\theta}})\,\mathrm{d}\Sigma \tag{4.15}$$

But

$$\delta U_3 = \int_V \sum_i (\boldsymbol{\sigma}_i \cdot \delta e_i + \boldsymbol{\mu}_i \cdot \delta\mathbf{k}_i)\,\mathrm{d}V. \qquad \delta U_3{}^* = \int_V \sum_i (\delta\boldsymbol{\sigma}_i \cdot \mathbf{e}_i + \delta\boldsymbol{\mu}_i \cdot \mathbf{k}_i)\,\mathrm{d}V. \tag{4.16}$$

Comparing (4.14) and (4.16), and considering that $\delta\boldsymbol{\sigma}_i$ is of the same order is $\delta\mathbf{e}_i$, and $\delta\boldsymbol{\mu}_i$ is of the same order as $\delta\mathbf{k}_i$, if becomes clear that $d(s, s')$ is of higher order than $\sqrt{|\delta F|}$. Equation (2.12) becomes then,

$$d^2(s, s_a') \leqslant |\delta F| + |\delta_a F| \tag{4.17}$$

for very small values of the distances between s and s'.

This means that the distance between s and s_a' tends to zero if the differences between the values of the total |potential / complementary| energy corresponding to s and s', and between s_a and s_a', also tend to zero.

In the sequel, s will denote the three-dimensional solution and s_a' the B_2-image of the two-dimensional solution. For understandable reasons, s will be called the exact solution and s_a' the approximate solution.

5. THE THREE-DIMENSIONAL MAGNITUDES AS FUNCTIONS OF THE THICKNESS

Laws of variation of the three-dimensional magnitudes in terms of the thickness will be sought now, having in mind that the corresponding two-dimensional magnitudes must not change when the thickness tends to zero.

The determination of such laws requires the definition of operator B_2 or, what is the same, the choice of the expressions for the three-dimensional |strains / stresses| and the three-dimensional |displacements / tractions| in terms of the corresponding two-dimensional magnitudes.

It is well known that many algebraic difficulties can be removed if the shell is assumed thin, i.e. if the magnitudes λ_α are assumed equal to unity. Thinness will be admitted in the sequel.

Consider a conventional shell, i.e. a shell with coefficients $\psi = \chi = \phi = 0$ and let t_0 be its thickness which, for sake of simplicity, is assumed constant all over the middle surface.

Let the expression for the three-dimensional $\begin{vmatrix}\text{strains}\\\text{stresses}\end{vmatrix}$ be the following:

$$e_{\alpha\beta} = e_{\beta\alpha} = \tfrac{1}{2}[E_{\alpha\beta} + E_{\beta\alpha} + \xi_3(\mathbf{K}_\alpha \wedge \mathbf{a}_3 \cdot \mathbf{a}_\beta + \mathbf{K}_\beta \wedge \mathbf{a}_3 \cdot \mathbf{a}_\alpha)] \quad\Big|\quad \sigma_{\alpha\beta} = \sigma_{\beta\alpha} = \frac{N_{\alpha\beta}}{t_0} + 12\mathbf{M}_\alpha \wedge \mathbf{a}_3 \cdot \mathbf{a}_\beta \frac{\xi_3}{t_0^{\,3}} \tag{5.1}$$

$$e_{\alpha 3} + e_{3\alpha} = E_{\alpha 3} \quad\Big|\quad \sigma_{\alpha 3} = \frac{N_{\alpha 3}}{t_0}\left[\frac{3}{2} - 6\left(\frac{\xi_3}{t_0}\right)^2\right] \tag{5.2}$$

$$e_{3\alpha} = e_{\alpha 3} \quad\Big|\quad \sigma_{3\alpha} = \sigma_{\alpha 3} \tag{5.3}$$

$$e_{33} = -\frac{\lambda_0(e_{11} + e_{22})}{2\mu_0} \quad\Big|\quad \sigma_{33} = 0 \tag{5.4}$$

We observe that equation (5.4 left) is equivalent to $\sigma_{33} = 0$.

From (5.1–5.4) and the invariance of internal work, we obtain the following expressions for the two-dimensional $\begin{vmatrix}\text{stresses}\\\text{strains}\end{vmatrix}$ in terms of the three-dimensional ones:

$$N_{\alpha\beta} = \int_{-t_0/2}^{t_0/2} \sigma_{\alpha\beta}\, d\xi_3 \quad\Big|\quad E_{\alpha\beta} + E_{\beta\alpha} = \frac{1}{t_0}\int_{-t_0/2}^{t_0/2} (e_{\alpha\beta} + e_{\beta\alpha})\, d\xi_3 \tag{5.5}$$

$$N_{\alpha 3} = \int_{-t_0/2}^{t_0/2} \sigma_{\alpha 3}\, d\xi_3 \quad\Big|\quad E_{\alpha 3} = \frac{1}{t_0}\int_{-t_0/2}^{t_0/2} \left[\frac{3}{2} - 6\left(\frac{\xi_3}{t_0}\right)^2\right](e_{\alpha 3} + e_{3\alpha})\, d\xi_3 \tag{5.6}$$

$$\mathbf{M}_\alpha = \int_{-t_0/2}^{t_0/2} \mathbf{a}_3 \xi_3 \wedge \boldsymbol{\sigma}_\alpha\, d\xi_3. \quad\Big|\quad \mathbf{K}_\alpha \wedge \mathbf{a}_3 \cdot \mathbf{a}_\beta + \mathbf{K}_\beta \wedge \mathbf{a}_3 \cdot \mathbf{a}_\alpha = \frac{12}{t_0^{\,3}} \int_{-t_0/2}^{t_0/2} \xi_3(e_{\alpha\beta} + e_{\beta\alpha})\, d\xi_3. \tag{5.7}$$

We observe that $N_{\alpha 3}$ would cease to represent the resultant of stresses $\sigma_{\alpha 3}$ (equation 5.6 left) if strains $e_{\alpha 3}$ and $e_{3\alpha}$ were not assumed uniformly distributed in the thickness (equation 5.2 left).

We observe also that, by virtue of the thinness assumption and equations (4.1) and (4.2), $N_{12} = N_{21}$ and $M_{11} = -M_{22}$. This is important for the derivation of equations (5.5–5.7 right).

The expression for the two-dimensional coefficients can be obtained also from (5.1–5.4), following the standard procedure described in Section 4, i.e. by using equations (4.12), (3.9) and (3.10).

We obtain:

$$\bar{\mu} = \mu_0 t_0 \quad\Big|\quad \bar{\mu} = \mu_0 t_0 \tag{5.8}$$

$$\bar{\lambda} = \lambda_0 t_0 \quad\Big|\quad \bar{\lambda} = \frac{2\mu_0 \lambda_0}{2\mu_0 + \lambda_0} t_0 \tag{5.9}$$

$$\bar{\nu} = \frac{\mu_0 t_0}{2} \quad\Big|\quad \bar{\nu} = \frac{5}{12}\mu_0 t_0 \tag{5.10}$$

$$\bar{\psi} = \frac{\mu_0 t_0^{\,3}}{12} \qquad \bar{\psi} = \frac{\mu_0 t_0^{\,3}}{12} \tag{5.11}$$

$$\bar{\chi} = (\mu_0 + \lambda_0)\frac{t_0^{\,3}}{12} \qquad \bar{\chi} = \frac{2\mu_0 + 3\lambda_0}{2\mu_0 + \lambda_0}\frac{\mu_0 t_0^{\,3}}{12} \tag{5.12}$$

$$\bar{\pi} = \bar{\phi} = \bar{\tau} = 0. \qquad \bar{\pi} = \bar{\phi} = \bar{\tau} = 0. \tag{5.13}$$

Index $_0$ indicates that the two-dimensional coefficients refer to the shell with thickness t_0.

Assume now that the thickness of the shell tends to zero. The equations (5.8–5.13) clearly show that what was already mentioned before, i.e. that the ratio between the bending stiffness coefficients $\bar{\psi}$ and $\bar{\chi}$ and the stiffness coefficients $\bar{\mu}$, $\bar{\lambda}$ and $\bar{\nu}$ also tend to zero. Indeed, the first ones are proportional to t^3 and the last ones to t. By other words, a conventional shell behaves more and more like a membrane as its thickness tends to zero.

The two-dimensional coefficients cannot be kept constant therefore, as the thickness tends to zero, unless non-vanishing couple-stresses are admitted in the shell, i.e. unless the three-dimensional coefficients ψ, χ and/or ϕ cease to be zero.

Admitting non-vanishing couple-stresses, equations (5.1–4) must be replaced by

$$e_{\alpha\beta} = e_{\beta\alpha} = \tfrac{1}{2}[E_{\alpha\beta} + E_{\beta\alpha} + \xi_3(\mathbf{K}_\alpha \wedge \mathbf{a}_3 \cdot \mathbf{a}_\beta + \mathbf{k}_\beta \wedge \mathbf{a}_3 \cdot \mathbf{a}_\alpha)] \qquad \sigma_{\alpha\beta} = \sigma_{\beta\alpha} = \frac{N_{\alpha\beta}}{t} + 12\eta \mathbf{M}_\alpha \wedge \mathbf{a}_3 \cdot \mathbf{a}_\beta \frac{\xi_3}{t^3} \tag{5.14}$$

$$e_{\alpha 3} + e_{3\alpha} = E_{\alpha 3} \qquad \sigma_{\alpha 3} = \frac{N_{\alpha 3}}{t}\left[\frac{3}{2} - 6\left(\frac{\xi_3}{t}\right)^2\right] \tag{5.15}$$

$$e_{3\alpha} = \frac{\rho - \mu + \omega(\rho + \mu)}{\rho + \mu + \omega(\rho - \mu)} e_{\alpha 3} \qquad \sigma_{3\alpha} = \omega \sigma_{\alpha 3} \tag{5.16}$$

$$e_{33} = -\frac{\lambda(e_{11} + e_{22})}{2\mu} \qquad \sigma_{33} = 0 \tag{5.17}$$

$$k_{\alpha\beta} = K_{\alpha\beta} \qquad \mu_{\alpha\beta} = (1 - \eta)\frac{M_{\alpha\beta}}{t} \tag{5.18}$$

$$k_{i3} = k_{3i} = 0. \qquad \mu_{i3} = \mu_{3i} = 0. \tag{5.19}$$

It is admitted in equations (5.14–18 right) that a part of the moment vectors $\mathbf{M}_\alpha$ is equilibrated by the stress couples and the remaining part by the couple-stresses. Coefficient η, which is assumed equal to one for $t = t_0$ and equal to zero for $t = 0$, takes care of such partition.

Equation (5.16 right) reflects the fact that, as it will be shown in Section 6, stresses $\sigma_{3\alpha}$ must remain bounded as t tends to zero, while stresses $\sigma_{\alpha 3}$ are of the order of $1/t$. Coefficient ω is thus assumed equal to one for $t = t_0$ and of the order of t^α where $\alpha \gg 1$.

In what concerns equation (5.16 left), it results from introducing $\sigma_{3\alpha} = \omega\sigma_{\alpha 3}$ in the stress–strain equations.

The new expressions for the two-dimensional $\left|\begin{matrix}\text{stresses}\\ \text{strains}\end{matrix}\right|$ which are consistent with (5.14–19) are

$$N_{\alpha\beta} = \int_{-t/2}^{t/2} \sigma_{\alpha\beta}\, d\xi_3 \qquad E_{\alpha\beta} + E_{\beta\alpha} = \frac{1}{t}\int_{-t/2}^{t/2} (e_{\alpha\beta} + e_{\beta\alpha})\, d\xi_3 \tag{5.20}$$

$$N_{\alpha 3} = \int_{-t/2}^{t/2} \sigma_{\alpha 3}\, d\xi_3 \qquad E_{\alpha 3} = \frac{1}{t}\int_{-t/2}^{t/2}\left[\frac{3}{2} - 6\left(\frac{\xi_3}{t}\right)^2\right](e_{\alpha 3} + e_{3\alpha})\, d\xi_3 \tag{5.21}$$

$$\mathbf{M}_\alpha = \int_{-t/2}^{t/2} (\boldsymbol{\mu}_\alpha + \mathbf{a}_3 \xi_3 \wedge \vec{\mathbf{a}}_\alpha)\, \mathrm{d}\xi_3 .$$

$$\mathbf{K}_\alpha \wedge \mathbf{a}_3 \cdot \mathbf{a}_\beta + \mathbf{K}_\beta \wedge \mathbf{a}_3 \cdot \mathbf{a}_\alpha = \int_{-t/2}^{t/2} \left[12\eta \frac{\xi_3}{t^3}(e_{\alpha\beta} + e_{\beta\alpha}) + \frac{1}{t}(1-\eta)(\mathbf{k}_\alpha \wedge \mathbf{a}_3 \cdot \mathbf{a}_\beta + \mathbf{k}_\beta \wedge \mathbf{a}_3 \cdot \mathbf{a}_\alpha)\right] \mathrm{d}\xi_3 . \quad (5.22)$$

The two-dimensional coefficients become

$$\bar{\mu} = \mu t \qquad\qquad \bar{\mu} = \mu t \quad (5.23)$$

$$\bar{\lambda} = \lambda t \qquad\qquad \bar{\lambda} = \frac{2\mu\lambda}{2\mu + \lambda} t \quad (5.24)$$

$$\bar{\nu} = \frac{1}{2} \frac{\omega\left[(1+\omega) + \dfrac{\mu}{\rho}(1-\omega)\right]}{(1+\omega)^2} \mu t \qquad\qquad \bar{\nu} = \frac{5}{3} \frac{\mu t}{\left[(1+\omega)^2 + \dfrac{\mu}{\rho}(1-\omega)^2\right]} \quad (5.25)$$

$$\bar{\chi} = (\mu + \lambda)\frac{t^3}{12} + \lambda t \qquad\qquad \bar{\chi} = \frac{1}{\dfrac{2\mu+\lambda}{2\mu+3\lambda}\dfrac{12}{\mu t^3}\eta^2 + \dfrac{(1-\eta)^2}{\lambda t}} \quad (5.26)$$

$$\bar{\psi} = \frac{\mu t^3}{12} + \psi t \qquad\qquad \bar{\psi} = \frac{1}{\dfrac{12}{\mu t^3}\eta^2 + \dfrac{(1-\eta)^2}{\psi t}} \quad (5.27)$$

$$\bar{\pi} = \bar{\phi} = \bar{\tau} = 0. \qquad\qquad \bar{\pi} = \bar{\phi} = \bar{\tau} = 0. \quad (5.28)$$

As it was to be expected, expressions (5.8–5.13) are obtained if $t = t_0$ and $\eta = \omega = 1$. Equating now the two-dimensional coefficients in (5.8–5.13) and (5.23–5.28), we obtain:

$$\mu = \mu_0 \frac{t_0}{t} \qquad\qquad \mu = \mu_0 \frac{t_0}{t} \quad (5.29)$$

$$\lambda = \lambda_0 \frac{t_0}{t} \qquad\qquad \lambda = \lambda_0 \frac{t_0}{t} \quad (5.30)$$

$$\rho = \frac{1-\omega}{1+\omega} \mu_0 t_0 \frac{1}{t} \qquad\qquad \rho = \mu_0 t_0 \frac{(1-\omega)^2}{4 - (1+\omega)^2} \frac{1}{t} \quad (5.31)$$

$$\psi = \frac{\mu_0 t_0^3}{12}\left(1 - \frac{t^2}{t_0^2}\right)\frac{1}{t} \qquad\qquad \psi = \frac{\mu_0 t_0^3}{12} \frac{(1-\omega)^2}{1 - \omega^2 \dfrac{t_0^2}{t^2}} \frac{1}{t} \quad (5.32)$$

$$\chi = (\mu_0 + \lambda_0)\frac{t_0^3}{12}\left(1 - \frac{t^2}{t_0^2}\right)\frac{1}{t} \qquad\qquad \chi = \frac{2\mu_0 + 3\lambda_0}{2\mu_0 + \lambda_0} \cdot \frac{\mu_0 t_0^3}{12} \cdot \frac{(1-\eta)^2}{1 - \eta^2 \dfrac{t_0^2}{t^2}} \frac{1}{t} \quad (5.33)$$

$$\phi = 0. \qquad\qquad \phi = 0. \quad (5.34)$$

These equations show that the three-dimensional coefficients are of the order of $1/t$.

The variation of the three-dimensional coefficients with the thickness will be completely defined only if η and ω are given as functions of the thickness. It is interesting to notice that, if we choose

$$\eta = \frac{t^2}{t_0^{\,2}} \tag{5.35}$$

the expression for ψ becomes the same in both approaches.

The invariance of internal work, together with equations (5.14–5.19), could give still, for the second approach, the expressions for the two-dimensional prescribed initial strains. In the sequel, however, we shall suppose such prescribed initial strains equal to zero.

In order that operator B_2 be completely defined, expressions for the three-dimensional |displacements / tractions| must also be given. Adopting

$$u_\alpha = U_\alpha + \xi_3 \mathbf{\Theta} \wedge \mathbf{a}_3 \cdot \mathbf{a}_\alpha \qquad \Big| \qquad \sigma_\alpha = \frac{N_\alpha}{t} + 12\eta \mathbf{M} \wedge \mathbf{a}_3 \cdot \mathbf{a}_\alpha \frac{\xi_3}{t^3} \tag{5.36}$$

$$u_3 = U_3 \qquad \Big| \qquad \sigma_3 = \frac{N_3}{t}\left[\frac{3}{2} - 6\left(\frac{\xi_3}{t}\right)^2\right] \tag{5.37}$$

$$\boldsymbol{\theta} = \mathbf{\Theta} \qquad \Big| \qquad \boldsymbol{\mu} = (1-\eta)\frac{\mathbf{M}}{t} \tag{5.38}$$

the following expressions for the |prescribed tractions / prescribed displacements| obtained from the invariance of the external work, are obtained

$$N_\alpha = \int_{-t/2}^{t/2} \sigma_\alpha \, d\xi_3 \qquad \Big| \qquad \bar{U}_\alpha = \frac{1}{t}\int_{-t/2}^{t/2} \bar{u}_\alpha \, d\xi_3 \tag{5.39}$$

$$\bar{N}_3 = \int_{-t/2}^{t/2} \bar{\sigma}_3 \, d\xi_3 \qquad \Big| \qquad \bar{U}_3 = \frac{1}{t}\int_{-t/2}^{t/2} \left[\frac{3}{2} - 6\left(\frac{\xi_3}{t}\right)^2\right]\bar{u}_3 \, d\xi_3 \tag{5.40}$$

$$\bar{\mathbf{M}} = \int_{-t/2}^{t/2} (\mathbf{a}_3 \xi_3 \wedge \bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{\mu}}) \, d\xi_3 . \qquad \Big| \qquad \bar{\mathbf{\Theta}} = \int_{-t/2}^{t/2} \left[12\eta \frac{\xi_3}{t^3} \mathbf{a}_3 \wedge \bar{\mathbf{u}} + \frac{1}{t}(1-\eta)\bar{\boldsymbol{\theta}}\right] d\xi_3 \tag{5.41}$$

We observe that equations (5.39–5.41 left) are the same as equations (4.3) and (4.4) if, of course, the thinness assumption is admitted.

The following expressions for the two-dimensional body force densities in the first approach can be obtained also from equations (5.36–5.38) and the invariance of external work:

$$\bar{\mathbf{F}} = \int_{-t/2}^{t/2} \bar{\mathbf{f}} \, d\xi_3 \tag{5.42}$$

$$\bar{\mathbf{G}} = \int_{-t/2}^{t/2} (\bar{\mathbf{g}} + \mathbf{a}_3 \xi_3 \wedge \bar{\mathbf{f}}) \, d\xi_3 \tag{5.43}$$

(the prescribed initial strains were assumed to vanish)

These expressions coincide also with those given by equations (4.5) and (4.6), once the shell is admitted to be thin.

Expressions for the three-dimensional body force densities in terms of the two-dimensional densities, consistent with equations $\left|\begin{matrix}(5.42)\text{ and }(5.43)\\(4.5)\text{ and }(4.6)\end{matrix}\right|$ are also required. We may choose

$$\bar{\mathbf{f}} = \frac{\bar{\mathbf{F}}}{t} + 12\gamma\bar{\mathbf{G}} \wedge \mathbf{a}_3 \frac{\xi_3}{t^3} \tag{5.44}$$

$$\bar{\mathbf{g}} = (1-\gamma)\frac{\bar{\mathbf{G}}}{t} \tag{5.45}$$

where γ plays a role similar to the role of η in equations (5.14) and (5.18).

Making $\gamma = t^3/t_0{}^3$, we obtain

$$\bar{\mathbf{f}} = \frac{\bar{\mathbf{F}}}{t} + 12\bar{\mathbf{G}} \wedge \mathbf{a}_3 \frac{\xi_3}{t_0{}^3} \tag{5.46}$$

$$\bar{\mathbf{g}} = \left(1 - \frac{t^3}{t_0{}^3}\right)\frac{\bar{\mathbf{G}}}{t}. \tag{5.47}$$

The important points about (5.46–5.47) is that the body force densities are of the order of $1/t$ and its first derivatives with respect to ξ_3 are bounded.

Similar expressions can be established for the tractions:

$$\bar{\boldsymbol{\sigma}} = \frac{\bar{\mathbf{N}}}{t} + 12\bar{\mathbf{M}} \wedge \mathbf{a}_3 \frac{\xi_3}{t_0{}^3} \tag{5.48}$$

$$\bar{\boldsymbol{\mu}} = \left(1 - \frac{t^3}{t_0{}^3}\right)\frac{\bar{\mathbf{M}}}{t}. \tag{5.49}$$

6. ORDER OF MAGNITUDE OF THE STRESSES

Before any convergence analysis is made, the order of magnitude of the three-dimensional stresses, strains and displacements is to be compared with the order of t.

Conclusions can be drawn from the following points:

(i) As the two-dimensional solution does not depend on the thickness, the energy associated with such solution remains constant, and thus bounded, as the thickness decreases indefinitely. We admit therefore that the energy associated with the three-dimensional solution also remains bounded as t tends to zero, and so remain the three-dimensional strains and displacements as well as the first derivatives of the displacements.†

(ii) By virtue of equations (5.29–34), the three-dimensional elastic coefficients are of the order of t^{-1}.

(iii) Considering that the three-dimensional body force and moment densitites, $\bar{\mathbf{f}}$ and $\bar{\mathbf{g}}$, vary with t according to equations (5.46) and (5.47), there follows that such densities are of the order of t^{-1}.

(iv) As the faces of the shell are co-ordinates surfaces with equations $\xi_3 = \pm t/2$, the stresses σ_{3i} and μ_{3i} must have the same values on the faces as the tractions σ_i and μ_i and, as such values are prescribed, stresses σ_{3i} and μ_{3i} are bounded for $\xi_3 = \pm t/2$.

† We mean, of course, by strains and displacements, both ordinary strains and displacements, as well as curvatures and rotations.

It may be concluded from the preceeding remarks that:

(a) All the derivatives of the strains with order higher than the first, and all the derivatives of the displacements with order higher than the second, are of the order of t.

(b) The stresses σ_{3i} and μ_{3i} together with all the strains and displacements, the first derivatives of the strains, the first and second derivatives of the displacements and the derivatives of the stresses with order higher than the first, remain bounded everywhere in the shell (are of the order of t^0).

(c) All the remaining stresses and the first derivatives of the stresses are of the order of $1/t$.

Such conclusions are summarized in Table 1.

Table 1. Orders of magnitude of stresses, strains and displacements

Order of derivatives	Magnitudes			
	Stresses		Strains	Displacements
	$\sigma_{\alpha i}$ and $\mu_{\alpha i}$	σ_{3i} and μ_{3i}		
0	t^{-1}	t^0	t^0	t^0
1	t^{-1}	t^{-1}	t^0	t^0
2	t^0	t^0	t	t^0
higher than 2	t^0	t^0	t	t

Let us justify the above conclusions.

In order to show that the stresses are of the order of $1/t$, it suffices to remember that the strains are bounded (see i) and the elastic coefficients are of the order of $1/t$ (see ii).

We are going to show next that the second derivatives of the displacements with order higher than the second are of the order of $1/t$.

Indeed, the equilibrium equations written in terms of the displacements (Navier's equations) form an elliptical system of the second order and, according to a general property[9] of the elliptical equations of this kind, the derivatives of order $p + 2$ of the unknowns are of the same order as the pth derivatives of the right-hand sides. As the unknowns are the displacements and the right-hand sides of the Navier's equations result from dividing the body force densities (order $1/t$, by virtue of iii) by an elastic coefficient (order $1/t$, by virtue of ii), there follows that the right-hand sides are bounded and, therefore, that the second derivatives of the displacements are also bounded. On the other hand, as all the derivatives of the body force densities are bounded, the derivatives of the displacements of order higher than the second are of the order of t.

Now, as the seond derivatives of the displacements are bounded, and the initial strains vanish, the first derivatives of the strains are bounded and the first derivatives of the stresses are of the order of $1/t$.† As the derivatives of the displacements with order higher than the second are of the order of t, so are the derivatives of the strains with order higher than the first, the corresponding stress derivatives being bounded.†

Finally, as the stresses σ_{3i} and μ_{3i} have bounded values for $\xi_3 = \pm t/2$ (see iv), and the first derivatives of the stresses are of the order of $1/t$, there follows that such stresses are bounded for any value of ξ_3 or, which is the same, everywhere in the shell.

† Because the elastic coefficients are of the order of $1/t$ (see ii).

7. CONVERGENCE ANALYSIS

The distance between s and s_a' depends on δF and $\delta_a F$ (see equations 4.17).

Let us consider first the expression of δF.

$$\delta F = \delta U_3 - \int_V (\bar{\mathbf{f}} \cdot \delta\mathbf{u} + \bar{\mathbf{g}} \cdot \delta\boldsymbol{\theta})\, \mathrm{d}V - \int_{\Sigma_1} (\bar{\boldsymbol{\sigma}} \cdot \delta\mathbf{u} + \bar{\boldsymbol{\mu}} \cdot \delta\boldsymbol{\theta})\, \mathrm{d}\Sigma \quad \Bigg| \quad \delta F = \delta U_3^* - \int_V \sum_i (\delta\boldsymbol{\sigma}_i \cdot \bar{\mathbf{e}}_i^0 + \delta\boldsymbol{\mu}_i \cdot \bar{\mathbf{k}}_i^0)\, \mathrm{d}V - \int_{\Sigma_2} (\delta\boldsymbol{\sigma} \cdot \bar{\mathbf{u}} - \delta\boldsymbol{\mu} \cdot \bar{\boldsymbol{\theta}})\, \mathrm{d}\Sigma \tag{7.1}$$

where

$$\delta U_3 = \int_V \sum_i (\boldsymbol{\sigma}_i \cdot \delta\mathbf{e}_i + \boldsymbol{\mu}_i \cdot \delta\mathbf{k}_i)\, \mathrm{d}V \quad \Big| \quad \delta U_3^* = \int_V \sum_i (\delta\boldsymbol{\sigma}_i \cdot \mathbf{e}_i + \delta\boldsymbol{\mu}_i \cdot \mathbf{k}_i)\, \mathrm{d}V. \tag{7.2}$$

δ denotes variations from the exact solution s to its B-image s', or, which is the same, to the allowed field which presents the same $\left|\begin{matrix}\text{displacements and strains on the middle surface}\\ \text{resultants and couples taken on the thickness}\end{matrix}\right|$ (see equation 4.1–5). Denoting by a prime the magnitudes corresponding to s', we can write

$$\mathbf{e}_\alpha(0) = \mathbf{e}_\alpha'(0) \quad \Big| \quad \mathbf{N}_\alpha = \mathbf{N}_\alpha' \tag{7.3}$$

$$\mathbf{k}_\alpha(0) = \mathbf{k}_\alpha'(0) \quad \Big| \quad \mathbf{M}_\alpha = \mathbf{M}_\alpha' \tag{7.4}$$

$$\mathbf{u}(0) = \mathbf{u}'(0) \quad \Big| \quad \mathbf{N} = \mathbf{N}' \tag{7.5}$$

$$\boldsymbol{\theta}(0) = \boldsymbol{\theta}'(0). \quad \Big| \quad M = \mathbf{M}'. \tag{7.6}$$

Expanding the $\left|\begin{matrix}\text{strains}\\ \text{stresses}\end{matrix}\right|$ corresponding to s and s' in a power series, we obtain

$$\mathbf{e}_\alpha(\xi_3) = \mathbf{e}_\alpha(0) + \xi_3\, \mathbf{e}_{\alpha,3}(0) + \cdots \quad \Big| \quad \boldsymbol{\sigma}_\alpha(\xi_3) = \boldsymbol{\sigma}_\alpha(0) + \xi_3\, \boldsymbol{\sigma}_{\alpha,3}(0) + \cdots \tag{7.7}$$

$$\mathbf{k}_\alpha(\xi_3) = \mathbf{k}_\alpha(0) + \xi_3\, \mathbf{k}_{\alpha,3}(0) + \cdots \quad \Big| \quad \boldsymbol{\mu}_\alpha(\xi_3) = \boldsymbol{\mu}_\alpha(0) + \xi_3\, \boldsymbol{\mu}_{\alpha,3}(0) + \cdots \tag{7.8}$$

$$\mathbf{e}_\alpha'(\xi_3) = \mathbf{e}_\alpha'(0) + \xi_3\, \mathbf{e}_{\alpha,3}'(0) + \cdots \quad \Big| \quad \boldsymbol{\sigma}_\alpha'(\xi_3) = \boldsymbol{\sigma}_\alpha'(0) + \xi_3\, \boldsymbol{\sigma}_{\alpha,3}'(0) + \cdots \tag{7.9}$$

$$\mathbf{k}_\alpha'(\xi_3) = \mathbf{k}_\alpha'(0) + \xi_3\, \mathbf{k}_{\alpha,3}'(0) + \cdots \quad \Big| \quad \boldsymbol{\mu}_\alpha'(\xi_3) = \boldsymbol{\mu}_\alpha'(0) + \xi_3\, \boldsymbol{\mu}_{\alpha,3}'(0) + \cdots \tag{7.10}$$

As the derivatives of the $\left|\begin{matrix}\text{strains}\\ \text{stresses}\end{matrix}\right|$ with order higher than the first are $\left|\begin{matrix}\text{of the order of } t\\ \text{bounded}\end{matrix}\right|$, the terms ommited in the expressions (7.7) and (7.8) are certainly of higher order than the constant term and the linear term.

In what concerns expressions (7.9) and (7.10), it must be remembered that s' belongs to X' and that all the fields belonging to X' can be expressed by equations (5.14–19). Therefore, the derivatives of order higher than the first of $\left|\begin{matrix}e_{\alpha i}' \text{ and } k_{\alpha i}'\\ s_{\alpha i}' \text{ and } \mu_{\alpha i}'\end{matrix}\right|$ vanish.

†[Introducing (7.7–7.10 right) into (4.1) and (4.2), (with $\lambda_\alpha = 0$), we obtain

$$\left.\begin{aligned} \mathbf{N}_\alpha &= \boldsymbol{\sigma}_\alpha(0)t && (7.11)\\ \mathbf{M}_\alpha &= \boldsymbol{\mu}_\alpha(0)t + \mathbf{a}_3 \wedge \boldsymbol{\sigma}_{\alpha,3}'(0)\frac{t^3}{12} && (7.12)\\ \mathbf{N}_\alpha' &= \boldsymbol{\sigma}_\alpha'(0)t && (7.13)\\ \mathbf{M}_\alpha' &= \boldsymbol{\mu}_\alpha'(0)t + \mathbf{a}_3 \wedge \boldsymbol{\sigma}_{\alpha,3}'(0)\frac{t^3}{12} && (7.14) \end{aligned}\right\} + \text{higher order terms.}$$

† The part of the text within brackets concerns exclusively the complementary energy method.

As the first derivatives of the stresses are of the order of $1/t$ (see Section 6 and equation 5.14–19), there follows that (7.12) and (7.14) can be transformed into

$$\mathbf{M}_\alpha = \boldsymbol{\mu}_\alpha(0)t + \text{higher order terms} \tag{7.15}$$

$$\mathbf{M}_\alpha' = \boldsymbol{\mu}_\alpha'(0)t + \text{higher order terms.} \tag{7.16}$$

Equations (7.3) and (7.4) (right) are thus equivalent to

$$\boldsymbol{\sigma}_\alpha(0) = \boldsymbol{\sigma}_\alpha'(0) \tag{7.3$'$}$$

$$\boldsymbol{\mu}_\alpha(0) = \boldsymbol{\mu}_\alpha'(0) \tag{7.4$'$}$$

if higher order terms are neglected.]

Subtracting (7.9) from (7.7) and (7.10) from (7.8), and considering $\left|\begin{array}{c}\text{(7.3) and (7.4)}\\ \text{(7.3}'\text{) and (7.4}'\text{)}\end{array}\right|$, there results

$$\begin{array}{l|l} \delta\mathbf{e}_\alpha = \xi_3\,\delta\mathbf{e}_{\alpha,3}(0) + \cdots & \delta\boldsymbol{\sigma}_\alpha = \xi_3\,\delta\boldsymbol{\sigma}_{\alpha,3}(0) + \cdots \end{array} \tag{7.17}$$

$$\begin{array}{l|l} \delta\mathbf{k}_\alpha = \xi_3\,\delta\mathbf{k}_{\alpha,3}(0) + \cdots & \delta\boldsymbol{\mu}_\alpha = \xi_3\,\delta\boldsymbol{\mu}_{\alpha,3}(0) + \cdots \end{array} \tag{7.18}$$

Considering (7.17) and (7.18), and neglecting terms of higher order, we can write

$$\begin{array}{l|l} \displaystyle\int_{-t/2}^{t/2} (\boldsymbol{\sigma}_\alpha \cdot \delta\mathbf{e}_\alpha + \boldsymbol{\mu}_\alpha \cdot \delta\mathbf{k}_\alpha)\,\mathrm{d}\xi_3 & \displaystyle\int_{-t/2}^{t/2} (\delta\boldsymbol{\sigma}_\alpha \cdot \mathbf{e}_\alpha + \delta\boldsymbol{\mu}_\alpha \cdot \mathbf{k}_\alpha)\,\mathrm{d}\xi_3 \\ \displaystyle\qquad \simeq \int_{-t/2}^{t/2} \xi_3\,\delta\mathbf{e}_{\alpha,3}(0)\,\mathrm{d}\xi_3 . & \displaystyle\qquad \simeq \int_{-t/2}^{t/2} \xi_3\,\delta\boldsymbol{\sigma}_{\alpha,3}(0)\cdot\mathbf{e}_\alpha\,\mathrm{d}\xi_3 . \end{array} \tag{7.19}$$

Therefore, as the $\left|\begin{array}{c}\text{strains}\\ \text{stresses}\end{array}\right|$ and the first derivatives of the $\left|\begin{array}{c}\text{strains are bounded}\\ \text{stresses are of the order of } 1/t\end{array}\right|$ (see Section 6 and equations 5.14–5.19),

$$\begin{array}{l|l} \displaystyle\int_{-t/2}^{t/2} (\boldsymbol{\sigma}_\alpha \cdot \delta\mathbf{e}_\alpha + \boldsymbol{\mu}_\alpha \cdot \delta\mathbf{k}_\alpha)\,\mathrm{d}\xi_3 = 0(t). & \displaystyle\int_{-t/2}^{t/2} (\delta\boldsymbol{\sigma}_\alpha \cdot \mathbf{e}_\alpha + \delta\boldsymbol{\mu}_\alpha \cdot \mathbf{k}_\alpha)\,\mathrm{d}\xi_3 = 0. \end{array} \tag{7.20}$$

As, on the other hand, by virtue of the stresses σ_{3i} and μ_{3i} and the corresponding strains being bounded, both in the exact solution and in its B-image (see Section 6 and equations 5.14–5.19).

$$\begin{array}{c|c} \boldsymbol{\sigma}_3 \cdot \delta\mathbf{e}_3 + \boldsymbol{\mu}_3 \cdot \delta\mathbf{k}_3 & \delta\boldsymbol{\sigma}_3 \cdot \mathbf{e}_3 + \delta\boldsymbol{\mu}_3 \cdot \mathbf{k}_3 \end{array} \tag{7.21}$$

is bounded, so that we can write

$$\begin{array}{l|l} \displaystyle\delta U_3 = \int_V \sum_i (\boldsymbol{\sigma}_i \cdot \delta\mathbf{e}_i + \boldsymbol{\mu}_i \cdot \delta\mathbf{k}_i)\,\mathrm{d}V = 0(t). & \displaystyle\delta U_3{}^* = \int_V \sum_i (\delta\boldsymbol{\sigma}_i \cdot \mathbf{e}_i + \delta\boldsymbol{\mu}_i \cdot \mathbf{k}_i)\,\mathrm{d}V = 0(t). \end{array} \tag{7.22}$$

The same kind of reasoning can be repeated for the other terms in δF, leading to the conclusion that δF is of the order of t.

Let us consider now $\delta_a F$.

s_a is simply a field which $\left|\begin{array}{c}\text{compatibilizes the same incompatibilities}\\ \text{equilibrates the same external forces}\end{array}\right|$ as the exact solution. Such a field can easily be constructed. We can take for instance as three-dimensional $\left|\begin{array}{c}\text{strains and displacements}\\ \text{stresses and tractions}\end{array}\right|$

$$\mathbf{e}_\alpha = \mathbf{E}_\alpha \qquad \Bigg| \qquad \boldsymbol{\sigma}_\alpha = \frac{\mathbf{N}_\alpha}{t} \tag{7.23}$$

$$\mathbf{k}_\alpha = \mathbf{K}_\alpha \qquad \Bigg| \qquad \boldsymbol{\mu}_\alpha = \frac{\mathbf{M}_\alpha}{t} \tag{7.24}$$

$$\mathbf{u} = \mathbf{U} \qquad \Bigg| \qquad \boldsymbol{\mu} = \frac{\mathbf{M}}{t} \tag{7.25}$$

$$\boldsymbol{\theta} = \boldsymbol{\Theta} \qquad \Bigg| \qquad \boldsymbol{\sigma} = \frac{\mathbf{N}}{t} \tag{7.26}$$

where $\left|\begin{matrix}\mathbf{E}_\alpha, \mathbf{K}_\alpha, \mathbf{U} \text{ and } \boldsymbol{\Theta} \\ \mathbf{N}_\alpha, \mathbf{M}_\alpha, N \text{ and } \mathbf{M}\end{matrix}\right|$ are the two-dimensional $\left|\begin{matrix}\text{strains and displacements} \\ \text{stresses and tractions}\end{matrix}\right|$ corresponding to the approximate solutions.

Introducing (7.23–7.26) into the three-dimensional $\left|\begin{matrix}\text{strain–displacement} \\ \text{equilibrium}\end{matrix}\right|$ equations, we obtain

$$\mathbf{e}_i = \mathbf{U}_{,i} + \mathbf{a}_i \wedge \boldsymbol{\Theta} \qquad \Bigg| \qquad \frac{1}{t}\sum_\alpha (h_{\bar\alpha}\mathbf{N}_\alpha)_{,\alpha} + (h_1 h_2 \boldsymbol{\sigma}_3)_{,3} + h_1 h_2 \bar{\mathbf{f}} = \mathbf{0} \tag{7.27}$$

$$\mathbf{k}_i = \boldsymbol{\Theta}_{,i}. \qquad \Bigg| \qquad \frac{1}{t}\sum_\alpha [(h_{\bar\alpha}\mathbf{M}_\alpha)_{,\alpha} + h_1 h_2 \mathbf{a}_\alpha \wedge \mathbf{N}_\alpha] + (h_1 h_2 \mu_3)_{,3} + h_1 h_2 \mathbf{a}_3 \wedge \boldsymbol{\sigma}_3 + h_1 h_2 \bar{\mathbf{g}} = 0 \tag{7.28}$$

As the two-dimensional $\left|\begin{matrix}\text{strain–displacement} \\ \text{equilibrium}\end{matrix}\right|$ equations are satisfied and the shell is thin, equations (7.27) and (7.28) reduce simply to

$$\mathbf{e}_3 = \mathbf{a}_3 \wedge \boldsymbol{\Theta} \qquad \Bigg| \qquad \boldsymbol{\sigma}_{3,3} = \frac{\bar{\mathbf{F}}}{t} - \bar{\mathbf{f}} \tag{7.29}$$

$$\mathbf{k}_3 = \mathbf{0} \qquad \Bigg| \qquad \boldsymbol{\mu}_{3,3} - \mathbf{a}_3 \wedge \boldsymbol{\sigma}_3 = \frac{\bar{\mathbf{G}}}{t} - \bar{\mathbf{g}} \tag{7.30}$$

Equations (7.29) and (7.30) permit to obtain $\left|\begin{matrix}\mathbf{e}_3 \text{ and } \mathbf{k}_3 \\ \boldsymbol{\sigma}_3 \text{ and } \boldsymbol{\mu}_3\end{matrix}\right|$ such that the three-dimensional $\left|\begin{matrix}\text{strain–displacements} \\ \text{equilibrium}\end{matrix}\right|$ equations are fulfilled.

Such vectors and the ones expressed by (7.23) and (7.24) define completely the field s_a.

The evaluation of $\delta_a F$ can be made exactly as the evaluation of δF, and the conclusion can be drawn that

$$\delta_a F = 0(t). \tag{7.31}$$

Then, by virtue of (4.17),

$$d(s, s_a') = 0(t^{1/2}) \tag{7.32}$$

and this means that the exact solution converges to the approximate solution as the thickness tends to zero.

8. CONCLUSIONS

The present paper may be seen under two different points of view: either as just a paper on shells, or as one step more towards the constitution of what may be called a mathematical theory of structures.

Let us consider the first point of view.

A comprehensive discussion on the foundations of the theory of shells was recently presented by Naghdi[10] who indicates as the main problems of the theory of shells:

(a) The development of a two-dimensional theory.

(b) The development of a scheme or a systematic procedure for estimating the error involved in the use of such theory.

Naghdi mentions that his paper is mainly concerned with the first problem, and not with the second, for which an explicit answer has not been available.

The present paper can be seen as a contribution for the solution of the second problem. It provides indeed a simple but powerful method for testing the efficiency and consistency of any particular theory of shells whose equations can be established according to one of two dual procedures. Such procedures which, in the text, are called the potential energy and the complementary energy approaches, do not represent really more than a formalization of the energy approaches popularized by Reissner[11] in his paper about beams, plates and shells.

The method was exemplified by applying it to the theory of thin shells. Other approximations can be tested in a similar way and the order of magnitude of the distance between the approximate and the exact solutions can tell us about their efficiency. In the case of the thin shell approximation such distance was seen to be of the order of $\sqrt{t}$.

Consistency requires that all the terms in δF and $\delta_a F$ are of the same order. If they are not, this means that the approximation of the magnitudes involved in the higher order terms is unnecessarily high, or that the approximation of the remaining magnitudes is too low.

For instance, if the distribution assumed for stresses σ_{33} is such that equilibrium with the tractions actually acting on the faces is ensured, i.e. if stresses σ_{33} take the prescribed values for $\xi_3 = \pm t/2$, then, the corresponding terms in δU are of the order of t^3 and not of the order of t. The global accuracy is not increased however if the expressions for the remaining stresses are the ones given in the paper, because the corresponding terms in δU are of the order of t.

As the proposed test is based in examining what happens when the thickness tends to zero, it may be argued that it seems not very logical to examine the value of a theory intended for thick shells, for instance, by seeing what happens when the thickness decreases indefinitely.

The determination of the order of magnitude of the distance in terms of the thickness gives however a good way of evaluating the speed of convergence, and the higher the speed of convergence, the higher the values of the thickness for which the same distance between the approximate and the exact solution, i.e. the same error, is reached.

In what concerns the second point of view, it must be remembered that convergence analyses have been made for discrete theories generated by the finite element technique, and convergence theorems were presented for the whole theory of structures and even for variational methods in general[3, 5, 7, 8].

On the other hand, the role of such theorems in the theory of structures was recognized and the way in which they should be applied for the justification of continuous theories was indicated.

The idea of such application was however not materialized before the present paper, and the fact that the same kind of analysis which has been successfully applied to discrete models is now applied to continuous ones represents a step forward towards the construction of a general mathematical theory of elastic structures.

REFERENCES

1. W. T. Koiter (editor) *Proc. 1st. IUTAM Symp. on the Theory of Thin Shells.* Delft (1959). North-Holland (1960).
2. F. I. Niordson (editor) *Proc. 2nd. IUTAN Symp. on the Theory of Thin Shells.* Copenhagen (1967). Springer (1969).
3. E. R. A. Oliveira, The Convergence theorems and their role in the theory of structures, *Proc. IUTAM Symp. on High Speed Computing of Elastic Structures.* Liège (1970).
4. E. Kroner (editor) *Proc. IUTAM Symp. on Mechanics of Generalized Continua.* Springer (1968).
5. E. R. A. Oliveira, A general theory of variational methods with application to finite elements. Lisbon, December (1972).
6. A. E. Taylor, *Introduction to Functional Analysis.* Wiley (1958).
7. J. T. Oden and E. R. A. Oliveira (editors) *Lectures on Finite Element Methods in Continuum Mechanics.* UAH Press (1972).
8. E. R. A. Oliveira, *Lectures on the Foundations of the Theory of Structures.* Instituto Gulbenkian de Ciência, Oeiras (1972).
9. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. II. International (1962).
10. P. M. Naghdi, The theory of shells and plates. *Handbuch der Physik*, Vol. VI a/2. Springer (1972).
11. E. Reissner, A new derivation of the equations for the deformation of elastic shells, *Am. J. Math.* **63** 177–184 (1941); Variational considerations for elastic beams and shells. *ASCE Proc.* (*Eng. Mech. Div.*) *EM1.* February (1962); On the form of variationally derived shell equations, *J. Appl. Mech.* **31**, 233–238 (1964).

Абстракт — Подобным образом к этому, что происходит в методе конечного элемента, можно принять понятие сходимости, для подтверждения права пользования принципом виртуальной работы и вариационными теоремами, в целью определения уравнений теории оболочек.

Оказывается, что двухмерное решение приближается более и более к трехмерному, если толщина стремится к нулью, при условии, что относительные значения коэффициентов изгибной и мембранной жесткости не изменяются, когда оболочка принимает вид более тонкой.

Такое условие можно учитывать только для случая обобщенной оболочки, т. е. если предполагается, что моментные напряжения не исчезают.

Анализ определяет верхный предел относительно размера промежутка между точным решением и приближенным. Это является мощным орудием для исследования эффективности и совместности любой частной теории оболочек.